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Subword complexity of uniform D0L words over finite groups

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ABSTRACT

We deal with the subword complexity of uniform D0L words obtained from group substitutions. Our main interest is whether the subword complexity is "almost proportional" to the length of the factor. We find necessary and sufficient conditions for that. For some cases we show that this is impossible.

1. Introduction

Let Σ be a finite alphabet, and Σ^* the set of all finite words over Σ. A *non-erasing substitution* over Σ is a function $\xi : \Sigma \to \Sigma^*$, which associates with each letter $\alpha \in \Sigma$ a word $\xi(\alpha) \in \Sigma^*$ of *length* $|\xi(\alpha)| > 0$. In particular, substitutions are used in the definition of grammars, which are in their turn a standard mean to define languages.

A substitution ξ is *uniform* if all the words $\xi(\alpha)$, $\alpha \in \Sigma$, are of the same length $|\xi(\alpha)| \geq 2$. Any substitution ξ induces a map from Σ^* to Σ^* by putting $\xi(w) = \xi(w_1)\xi(w_2)\dots\xi(w_n)$ for $w = w_1w_2\dots w_n$, and likewise defines a map from $\Sigma^{\mathbb{N}}$ to $\Sigma^{\mathbb{N}}$, also denoted by ξ. Let $\alpha \in \Sigma$, and assume that $\xi(\alpha)$ begins with α and $|\xi(\alpha)| > 1$. Then for each k the word $\xi^{k+1}(\alpha)$ begins with $\xi^k(\alpha)$. Let $x = \xi^\infty(\alpha) \in \Sigma^{\mathbb{N}}$ be the limit of $\left(\xi^k(\alpha)\right)_{k=1}^{\infty}$. Clearly, x is a fixed point of ξ, also known as a *D0L word* (cf. [3–7,12]).

Example. Define ξ over $\Sigma = \{0, 1\}$ by $0 \to 01$ and $1 \to 10$. Then $x = \xi^\infty(0) = 0110100110010110\dots$, which is the well-known Thue–Morse sequence (cf. [14,18,19]). One can show that $x_i = 1$ if and only if the number of 1's in the binary expansion of $i - 1$ is odd.

For a given word $x \in \Sigma^{\mathbb{N}}$, a *factor* of length n in x is a word $y = y_1y_2\dots y_n \in \Sigma^n$, for which there exists an $i \geq 0$ such that $x_{i+j} = y_j$ for $1 \leq j \leq n$. The function of subword complexity, $f(n)$, counts the number of factors of length n in x. Subword complexity functions are studied from various viewpoints. For instance, an interesting open question is which functions from $\mathbb{N}$ to $\mathbb{N}$ are subword complexity functions. (See [8] for a survey of known results regarding this question.) The function f provides the most basic and classical version of the subword complexity of an infinite word, but other complexity functions for finite and infinite words have been defined and studied, for instance, by Ferenczi and Kása [9].

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The subword complexity of D0L words was studied by Cassaigne [2], who also considered the number of so-called special and bi-special factors. The factors of a D0L word were studied, for instance, by Frid [12], who considered also the frequency of these factors. The subword complexity of a D0L word has also been studied, among others, by Ehrenfeucht and Rozenberg [3–7], Mossé [15] and Tapsoba [17]. For a survey of the area we refer to Allouche [1].

The growth rate of the subword complexity function of a D0L word is at most quadratic. Pansiot [16] showed that the function is in one of the families, $\Theta\left(n^2\right)$, $\Theta\left(n\log n\right)$, $\Theta\left(n\log\log n\right)$, $\Theta\left(n\right)$, or $\Theta\left(1\right)$, depending on the class of the substitution. Here we deal with uniform substitutions, and therefore the subword complexity of the D0L words presented in this paper is of linear growth.

On the other hand, symmetric morphisms, whose fixed point and complexity have been studied by Frid [10], often serve as examples of infinite words with needed properties. In particular, the words avoiding powers, abelian powers, and so on.

In this paper, we study the case where the alphabet Σ is a finite group G, and the substitution is related to the multiplication in the group. These substitutions generalize the symmetric morphisms. On the other hand, these substitutions give rise to marked uniform D0L words, whose subword complexity was also studied by Frid [11]. For example, the substitution generating the Thue–Morse sequence is a group substitution.

Group substitutions were introduced in [13]. We continue our study in [13] of the quantities

$$\liminf_{n\to\infty}\frac{f\left(n\right)}{n},\qquad \limsup_{n\to\infty}\frac{f\left(n\right)}{n},\tag{1.1}$$

where f is the subword complexity function of the D0L word formed by the group substitution. A close relation of group substitutions with another family of substitutions, that were termed auxiliary substitutions, was found in [13]. Hence we continue studying this family as well. This relation provides an easy algorithmic way for calculating the quantities in (1.1), which was also introduced in [13], and the most interesting objects turned out to be those group substitutions ξ for which $\xi\left(e\right)$ is a permutation of G, where e is the unit element of G.

Does there exist a group substitution for which the limits in (1.1) coincide? The answer turns out to be affirmative, and our first result gives necessary and sufficient conditions for that. Then, since the most interesting group substitutions seem to be those for which $\xi\left(e\right)$ is a permutation of G, we search for such substitutions which make the limits in (1.1) coincide. We did not find such a case; indeed, our results may hint that this cannot be the case (at least over finite groups). In fact, this may be proved in some cases.

In Section 2, we present the family substitutions we deal with, present their most basic properties and a few definitions which are used throughout the paper. Then, in Section 3 we state our conjecture and our main results, and in Sections 4 and 5 prove them.

2. Basic definitions and properties

2.1. Group substitutions

Let G be a finite group of order g. Denote by e its identity element. A *group substitution* of G is a substitution ξ of G of the form

$$\gamma\to\left(\gamma a_1\right)\left(\gamma a_2\right)\ldots\left(\gamma a_t\right),\quad \gamma\in G,$$

where $a_1, a_2, \ldots, a_t$ elements of G (not necessarily distinct), $a_1 = e$, and $t \geq 2$. Let $w = w_1w_2\ldots$ be the fixed point of the map induced by ξ on $G^{\mathbb{N}}$, with $w_1 = e$. Let f be the subword complexity of w. Obviously, the word w consists only of elements of the subgroup $\langle a_1, a_2, \ldots, a_t\rangle$, generated by $a_1, a_2, \ldots, a_t$. Hence, without loss of generality, we may assume that $G = \langle a_1, a_2, \ldots, a_t\rangle$. This assumption yields

Proposition 2.1. $f\left(1\right) = g$.

Define a sequence $(z_i)_{i=0}^{\infty}$ by $z_i = w_{i+1}$ for $i \geq 0$. The following proposition may be rephrased as the statement that this sequence is strongly t-multiplicative.

Proposition 2.2. *For each $a \geq 0$ and $b \in [0, t-1]$ we have $z_{at+b} = z_az_b$.*

2.2. An auxiliary family of substitutions

In order to study group substitutions, another family of substitutions is introduced; as we shall see, results on the latter family will imply results for the former (see also [13]). Let Σ be an arbitrary alphabet. An *auxiliary substitution* is a substitution τ of the form

$$\gamma\to b_1b_2\ldots b_{t-1}\varphi\left(\gamma\right),\quad \gamma\in\Sigma,\tag{2.1}$$

where $t\geq 2$ is an integer, $\varphi:\Sigma\to\Sigma$ a bijection, and $b_1, b_2, \ldots, b_{t-1}$ elements of Σ. Let $y = y_1y_2\ldots$ be the fixed point of the mapping of $\Sigma^{\mathbb{N}}$, induced by this substitution. Obviously, $y_1 = b_1$. Let h be the subword complexity of y. Since the empty word is the only word of length 0, we have $h(0) = 1$. The importance of h stems from Theorem 3.1 in [13], which states that, if $\Sigma = G$, $b_i = a_i^{-1} a_{i+1}$ for $1 \le i \le t-1$, and $\varphi(\gamma) = a_t^{-1}\gamma$, then the equality $f(n) = g \cdot h(n-1)$ holds for $n \ge 2$. In particular, we will be interested in the limits:

$$\underline{L} = \liminf_{n\to\infty} \frac{h(n)}{n}, \qquad \bar{L} = \limsup_{n\to\infty} \frac{h(n)}{n}.$$

The following proposition is straightforward.

Proposition 2.3. *Let $n \in \mathbb{N}$. Then:*

$$y_n = \begin{cases} b_{n \bmod t}, & t \nmid n, \\ \varphi\left(y_{n/t}\right), & t | n. \end{cases}$$

Definition 2.4. Let $\boldsymbol{\alpha} = (\alpha_i)_{i=1}^{l_\alpha - 1}$ and $\boldsymbol{\beta} = (\beta_i)_{i=1}^{l_\beta - 1}$ be two sequences of lengths $l_\alpha - 1$ and $l_\beta - 1$, respectively, over Σ. The *compound sequence* of $\boldsymbol{\alpha}$ over $\boldsymbol{\beta}$ is the sequence $\boldsymbol{\alpha} \odot \boldsymbol{\beta} = (\gamma_i)_{i=1}^{l_\alpha l_\beta - 1}$ of length $l_\alpha l_\beta - 1$, where

$$\gamma_i = \begin{cases} \beta_{i \bmod l_\beta}, & l_\beta \nmid i, \\ \alpha_{i/l_\beta}, & l_\beta | i. \end{cases}$$

Remark 2.5. If $\boldsymbol{\alpha}$ is the empty sequence then $\boldsymbol{\alpha} \odot \boldsymbol{\beta} = \boldsymbol{\beta}$, and if $\boldsymbol{\beta}$ is the empty sequence then $\boldsymbol{\alpha} \odot \boldsymbol{\beta} = \boldsymbol{\alpha}$. Also note that for three given sequences, $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, we have $\boldsymbol{\alpha} \odot (\boldsymbol{\beta} \odot \boldsymbol{\gamma}) = (\boldsymbol{\alpha} \odot \boldsymbol{\beta}) \odot \boldsymbol{\gamma}$. Therefore, for k given sequences σ_i, $1 \le i \le k$, we may write $\bigodot_{j=1}^{k} \sigma_j$ unambiguously.

Example 2.6. Suppose that $\Sigma = \mathbb{Z}_5$, and consider the sequences $(\alpha_i)_{i=1}^{5-1} = (2, 3, 4, 0)$ and $(\beta_i)_{i=1}^{4-1} = (1, 1, 1)$. The compound sequence is

$$(\gamma_i)_{i=1}^{20-1} = (1, 1, 1, 2, 1, 1, 1, 3, 1, 1, 1, 4, 1, 1, 1, 0, 1, 1, 1).$$

3. Main results

3.1. Necessary and sufficient conditions for the equality of limits

In [13] we observed that either both limits $\underline{L}$ and $\bar{L}$ are 0, or both are positive. The case where $\underline{L} = \bar{L} = 0$ is completely studied in [13], and turned to be true if and only if $c = 1$ and $\varphi(b_1) = b_1$. The more interesting case is when $\underline{L} = \bar{L} > 0$. We call a substitution τ *steady* in case $\underline{L} = \bar{L} > 0$. In [13, Theorem 3.3] we provided an algorithm for calculating $\underline{L}$ and $\bar{L}$, which states that if either $c \ne 1$ or $\varphi(b_1) \ne b_1$, then for an integer N and a set A defined as follows:

(1) If $c = t$, let $N = t - 1$ and $A = \{h(m)/m : 1 \le m \le t-1\}$.
(2) If $c < t$, let $N = (s+1)t - c$ and

$$A = \left\{ \frac{h(m)}{m} - \frac{t-c}{mt} : 1 \le m \le s \right\} \cup \left\{ \frac{ch(s) + (t-c)h(s+1)}{(s+1)t - c} \right\} \cup \left\{ \frac{h(m)}{m} : s+1 \le m \le 2c \right\}.$$

we have

$$\bar{L} = \max A = \max_{\lceil N/t \rceil \le m \le N} \frac{h(m)}{m},$$
$$\underline{L} = \min A = \min_{\lceil N/t \rceil \le m \le N} \frac{h(m)}{m}.$$

The following proposition follows from that Theorem in a straightforward manner.

Proposition 3.1. *If the substitution τ is steady, then $\underline{L} = \bar{L} = h(1)$.*

Proposition 3.1 is proven in Section 4. Out of [13, Theorem 3.3] and [13, Lemma 5.3] it also stems that, if τ is steady, then $h(n+1) - h(n) = h(1)$ for $n \in \mathbb{N}$. Another consequence of [13, Theorem 3.3] is that $h(1)$ is always an upper bound on $\underline{L}$. The following proposition deals with the case $\underline{L} = h(1)$.

Proposition 3.2. *We have $\underline{L} = h(1)$ if and only if there exist no sequences $\boldsymbol{\alpha} = (\alpha_i)_{i=1}^{l_\alpha - 1}$ and $\boldsymbol{\beta} = (\beta_i)_{i=1}^{l_\beta - 1}$, such that $(b_i)_{i=1}^{t-1} = \boldsymbol{\alpha} \odot \boldsymbol{\beta}$ and the sequence $\boldsymbol{\alpha}$ satisfies:*

(1) *$l_\alpha \ge 2$,*
(2) *$\alpha_1 = \alpha_{l_\alpha - 1}$,*
(3) *for each $i \in [1, l_\alpha - 2]$, either $\alpha_i = \alpha_1$ or $\alpha_{i+1} = \alpha_1$.*

Proposition 3.2 is proven in Section 4.

Example 3.3. Suppose that $\Sigma = C_5$, $t = 15$, $\varphi(\alpha) = \alpha + 1$, and

$$(b_i)_{i=1}^{15-1} = (1, 2, 4, 1, 2, 3, 1, 2, 4, 1, 2, 4, 1, 2).$$

Then $\underline{L} < h(1)$, because the sequence $(b_i)_{i=1}^{15-1}$ is the compound sequence of $(4, 3, 4, 4)$ over $(1, 2)$, and the sequence $(4, 3, 4, 4)$ satisfies the three conditions in Proposition 3.2.

Propositions 3.1 and 3.2 present necessary conditions for τ to be steady. The following theorem provides necessary and sufficient conditions for that. It will be convenient to introduce a definition first.

Definition 3.4. A sequence $\boldsymbol{\alpha} = (\alpha_i)_{i=1}^{l_\alpha - 1}$ is *appropriate* if and only if the following conditions hold:

(1) $l_\alpha \geq 3$,
(2) there exist $k, k' \in [1, l_\alpha - 1]$ with $k' \neq l_\alpha - k$, such that $\gcd(k, l_\alpha) = 1$ and

$$\alpha_i = \alpha_{(i+k) \bmod l_\alpha}, \quad i \in [1, l_\alpha - 1] \setminus \{l_\alpha - k, k'\},$$
$$\alpha_{k'} \neq \alpha_{(k'+k) \bmod l_\alpha},$$

(3) $\alpha_1 \neq \alpha_{l_\alpha - 1}$,

Theorem 3.5. *The substitution τ is steady if and only if there exist two sequences $\boldsymbol{\alpha} = (\alpha_i)_{i=1}^{l_\alpha - 1}$ and $\boldsymbol{\beta} = (\beta_i)_{i=1}^{l_\beta - 1}$ such that $(b_i)_{i=1}^{t-1} = \boldsymbol{\alpha} \odot \boldsymbol{\beta}$, the sequence $\boldsymbol{\alpha}$ is appropriate, and either the sequence $\boldsymbol{\beta}$ is empty or the substitution defined by*

$$\gamma \to \beta_1 \beta_2 \ldots \beta_{l_\beta - 1} \varphi(\gamma), \quad \gamma \in \Sigma \tag{3.1}$$

is steady.

Remark 3.6. The substitution in (3.1) is again an auxiliary substitution, and therefore we may use the theorem in a recursive way to determine whether τ is steady.

The conditions of the theorem above hint also that steady auxiliary substitutions are quite rare, especially for large alphabets and long substitutions.

Theorem 3.5 is proven in Section 4.

Example 3.7. Suppose $\Sigma = C_5$, $t = 5$, $\varphi(\gamma) = \gamma + 1$, and

$$(b_i)_{i=1}^{5-1} = (1, 2, 1, 2).$$

We may let $\boldsymbol{\alpha} = (1, 2, 1, 2)$ and $\boldsymbol{\beta}$ be the empty sequence, to acquire $(b_i)_{i=1}^{5-1} = \boldsymbol{\alpha} \odot \boldsymbol{\beta}$. For $k = 2$ and $k' = 4$, the sequence $\boldsymbol{\alpha}$ satisfies the second condition of Definition 3.4, and it obviously satisfies the other conditions as well. Thus, $\boldsymbol{\alpha}$ is an appropriate sequence, and since $\boldsymbol{\beta}$ is the empty sequence, it turns out of Theorem 3.5 that τ is steady.

The following example shows how Theorem 3.5 may be effectively applied.

Example 3.8. Suppose $\Sigma = C_5$, $t = 15$, $\varphi(\gamma) = \gamma + 1$, and

$$(b_i)_{i=1}^{15-1} = (1, 2, 1, 2, 3, 1, 2, 1, 4, 1, 2, 1, 2).$$

For $\boldsymbol{\alpha} = (3, 4)$ and $\boldsymbol{\beta} = (1, 2, 1, 2)$ we have $(b_i)_{i=1}^{15-1} = \boldsymbol{\alpha} \odot \boldsymbol{\beta}$. For $k = 1$ and $k' = 1$, all conditions of Definition 3.4 are satisfied, and hence the sequence $\boldsymbol{\alpha}$ is appropriate. The sequence $\boldsymbol{\beta}$ is non-empty, and the substitution defined in Theorem 3.5 is given by

$$\gamma \to 1212\varphi(\gamma), \quad \gamma \in \Sigma. \tag{3.2}$$

This is the substitution we dealt with in Example 3.7, where we found that this substitution is steady. Thus, τ is steady as well.

Adopting [13, Theorem 3.1], we may use Theorem 3.5 for the case of group substitutions, as presented in the following example. Note that we use multiplicative notation as we may use our results non-abelian groups as well.

Example 3.9. Let $G = \langle \zeta \rangle \simeq C_5$ for $\zeta = e^{2\pi i/5}$, $t = 5$, and $(a_i)_{i=1}^{5} = \left\langle 1, \zeta, \zeta^3, \zeta^4, \zeta \right\rangle$. According to [13, Theorem 3.1], for $(b_i)_1^4 = \left(\zeta, \zeta^2, \zeta, \zeta^2\right)$ and $\varphi(\gamma) = \zeta^4 \gamma$ we have

$$\liminf_{n\to\infty} \frac{f(n)}{n} = 5\underline{L}, \qquad \limsup_{n\to\infty} \frac{f(n)}{n} = 5\bar{L}.$$

The sequence $(b_i)_1^4 = \left(\zeta, \zeta^2, \zeta, \zeta^2\right)$ is similar to the one we dealt with in Example 3.7, and therefore Theorem 3.5 implies that $\underline{L} = \bar{L} = h(1)$. By [13, Theorem 3.1]:

$$\liminf_{n\to\infty} \frac{f(n)}{n} = \limsup_{n\to\infty} \frac{f(n)}{n}.$$

3.2. Covering of a group

Theorem 3.1 in [13] and Theorem 3.5 characterize the cases when

$$\liminf_{n\to\infty}\frac{f(n)}{n}=\limsup_{n\to\infty}\frac{f(n)}{n}. \tag{3.3}$$

In [13] we also observed that the case where $(a_i)_{i=1}^t$ is a permutation of G is very special. For example, some possible values of the two partial limits are obtained when $(a_i)_{i=1}^t$ is a permutation of G, but not for any other $(a_i)_{i=1}^t$ of length $t=g$. In this subsection we deal with the case when $(a_i)_{i=1}^t$ is a permutation of G.

We start with the case where G is abelian, and therefore Theorem 3.5 and [13, Theorem 3.1] require us to study the case where we have an appropriate sequence, $\boldsymbol{\gamma}=(\gamma_i)_{i=1}^{l_\gamma}$, and a subset $B\subseteq G$ which satisfy

$$\bigcup_{i=1}^{l_\gamma}\gamma_i B=G$$

where the union is disjoint. Note that, since $\boldsymbol{\gamma}$ is appropriate, according to the second condition in the definition of an appropriate sequence, there exist $k,k'\in\left[1,l_\gamma-1\right]$ such that the following subsequences are palindromic: $(\gamma_i)_{i=1}^{k'-1}$, $(\gamma_i)_{i=k'+1}^{l_\gamma-1}$, $(\gamma_i)_{i=1}^{(k'+k)\bmod l_\gamma-1}$, and $(\gamma_i)_{i=(k'+k)\bmod l_\gamma+1}^{n-1}$, (proven in Lemma 5.5, and note that the empty sequence is considered to be palindromic). This property is noticed in Examples 3.7 and 3.8.

Suppose in general that G is an abelian group, and we have

(1) an integer d, such that $d|g$ and $d>2$,
(2) a sequence $(\gamma_i)_{i=1}^d\in G^d$, such that $\gamma_1=e$,
(3) a set $B\subseteq G$ such that $|B|=g/d$, and

$$\bigcup_{i=1}^{d}\gamma_i B=G. \tag{3.4}$$

Put $\delta_i=\gamma_i^{-1}\gamma_{i+1}$ for each $i\in[1,d-1]$.

Proposition 3.10. *Let $k_1,k_2\in[1,d-1]$ be distinct integers. Suppose that $|\{\delta_i:1\le i\le d-1\}|=2$, and the subsequences $(\delta_i)_{i=1}^{k_j-1}$ and $(\delta_i)_{i=k_j+1}^{d-1}$ are palindromic for $j=1,2$. Then $B=\left(\delta_{k_2}^{-1}\delta_{k_1}\right)B$.*

Proposition 3.10 is proven in Section 5. In the following example we encounter such a case.

Example 3.11. Suppose that $G=\langle\zeta\rangle\simeq C_{21}$ for $\zeta=e^{2\pi i/21}$, and put $d=7$, and

$$(\gamma_i)_{i=1}^7=\left(1,\zeta,\zeta^9,\zeta^{17},\zeta^{18},\zeta^{19},\zeta^6\right),\qquad B=\left\{1,\zeta^7,\zeta^{14}\right\}.$$

Note that (3.4) holds in this case, and

$$(\delta_i)_{i=1}^{d-1}=\left(\zeta,\zeta^8,\zeta^8,\zeta,\zeta,\zeta^8\right).$$

For $k_1=2$ and $k_2=5$ the subsequences $(\delta_i)_{i=1}^{k_j-1}$ and $(\delta_i)_{i=k_j+1}^{d-1}$, for $j=1,2$, are actually (ζ), $(\zeta^8,\zeta,\zeta,\zeta^8)$, $(\zeta,\zeta^8,\zeta^8,\zeta)$, and (ζ^8), all of which are palindromic. Hence, the proposition states that $B=\left(\delta_5^{-1}\delta_2\right)B=\zeta^7B$, which is indeed satisfied.

The following theorem is a straightforward consequence of Proposition 3.10.

Theorem 3.12. *Suppose that G is a cyclic group of prime order. If $t=g$ and $(a_i)_{i=1}^t$ is a permutation of G, then*

$$\liminf_{n\to\infty}\frac{f(n)}{n}<\limsup_{n\to\infty}\frac{f(n)}{n}.$$

Theorem 3.12 is proven in Section 5. Proposition 3.10 may be used to prove that many other families of abelian groups do not possess this property, which make us state that an abelian group G which does not possess this property must be of a very complicated structure. Out of all abelian groups of order up to 3^5-1, there does not exist any which does not possess this property. Thus, we state the following conjecture.

Conjecture 3.13. *Suppose G is abelian. If $t=g$ and $(a_i)_{i=1}^t$ is a permutation of G, then*

$$\liminf_{n\to\infty}\frac{f(n)}{n}<\limsup_{n\to\infty}\frac{f(n)}{n}.$$

In fact, for all we know, Conjecture 3.13 may hold for non-abelian groups as well. However, we state it only in the abelian case as here various consequences of Proposition 3.10 seem to hint strongly that it indeed holds, whereas nothing but some supports the non-abelian case.

4. Proof of Theorem 3.5

4.1. Notations and proof of Proposition 3.1

We start by recalling a few notations and results from [13].

Definition 4.1. Let $(\alpha_i)_{i=1}^{m-1}$ be a sequence of length $m-1$ over Σ.

(1) For $d \geq 1$, the sequence is *d-periodic* if $\alpha_{i+d} = \alpha_i$ for $1 \leq i \leq m-1-d$.
(2) The *cyclicity* of $(\alpha_i)_{i=1}^{m-1}$ is the smallest divisor c of m for which the sequence is c-periodic.

Note that the cyclicity of the empty sequence is 1. From now on, we denote by c the cyclicity of $(b_i)_{i=1}^{t-1}$. In case $c < t$, let $s \in \mathbb{N} \cup \{0, \infty\}$ be the largest (finite or infinite) number for which the word $\left(\varphi^{-1}(b_c)\right)^s$ is a factor of y.

Throughout this section we suppose that $\bar{L} > 0$, which implies by [13, Prop. 2.9] that either $c \neq 1$ or $\varphi(b_1) \neq b_1$. Therefore, [13, Prop. 2.10] yields that $1 \leq s \leq 2c-1$, and hence we may define an integer N and a set A of rationals as follows:

(1) If $c = t$, let $N = t-1$ and $A = \{h(m)/m : 1 \leq m \leq t-1\}$.
(2) If $c < t$, let $N = (s+1)t - c$ and

$$A = \left\{\frac{h(m)}{m} - \frac{t-c}{mt} : 1 \leq m \leq s\right\} \cup \left\{\frac{ch(s) + (t-c)h(s+1)}{(s+1)t-c}\right\} \cup \left\{\frac{h(m)}{m} : s+1 \leq m \leq 2c\right\}.$$

According to [13, Theorem 3.3]:

$$\bar{L} = \max A = \max_{\lceil N/t \rceil \leq m \leq N} \frac{h(m)}{m},$$
$$\underline{L} = \min A = \min_{\lceil N/t \rceil \leq m \leq N} \frac{h(m)}{m}.$$

Moreover, throughout this section, for $n \in \mathbb{N}$ and $i \in [0, t-1]$, we view $B_i^{(n)}$ as the set of factors of length n, appearing in y starting at a location congruent to $i+1$ modulo t:

$$B_i^{(n)} = \left\{x_1 x_2 \ldots x_n \in \Sigma^n : \exists m \geq 0, x_j = y_{mt+i+j}, 1 \leq j \leq n\right\}. \tag{4.1}$$

The empty word is also a factor of y, and hence $B_i^{(0)} = \{\lambda\}$ for $0 \leq i \leq t-1$. Thus, every factor of length n belongs to at least one $B_i^{(n)}$. In particular, $h(n) = \left|\bigcup_{i=0}^{t-1} B_i^{(n)}\right|$. Note that $B_i^{(n)} \neq \emptyset$ for each $n \in \mathbb{N}$ and i, $0 \leq i \leq t-1$.

Proof of Proposition 3.1. Suppose that $\underline{L} = \bar{L} > 0$. Therefore,

$$\max_{\lceil N/t \rceil \leq m \leq N} \frac{h(m)}{m} = \bar{L} = \underline{L} = \min_{\lceil N/t \rceil \leq m \leq N} \frac{h(m)}{m},$$

and hence $h(m)/m = h(n)/n$ for $m, n \in [\lceil N/t \rceil, N]$. Suppose $c < t$. The above equality implies that $h(2c)/2c = h(2c+1)/(2c+1)$. Hence, $h(2c) = 2c(h(2c+1) - h(2c))$, and by [13, Lemma 5.11] we have $h(2c) = 2ch(1) - c$. Since $h(2c+1) - h(2c) \in \mathbb{Z}$, the equality $h(2c) = 2c(h(2c+1) - h(2c))$ yields that $2c \mid h(2c)$. Since $2c \nmid 2ch(1) - c$, this yields a contradiction, which means that $c = t$. Therefore, $N = t-1$ and $\lceil N/t \rceil = 1$, which implies that $h(1)/1 = h(m)/m$ for each $m \in [N/t, N]$, and hence $\underline{L} = \bar{L} = h(1)$. $\square$

The definition of the sets $B_i^{(n)}$ and Proposition 2.3 yield straightforwardly the following useful lemma.

Lemma 4.2. *Let $n \in [1, t]$, and $i, j \in [0, t-1]$ with $i \neq j$. Then $\left|B_i^{(n)} \cap B_j^{(n)}\right| \leq 1$.*

4.2. The case where $(b_i)_{i=1}^{t-1}$ is a compound sequence

The case where $(b_i)_{i=1}^{t-1}$ is a compound sequence is especially interesting. Suppose that there exist two sequences $\boldsymbol{\alpha} = (\alpha_i)_{i=1}^{l_\alpha - 1}$ and $\boldsymbol{\beta} = (\beta_i)_{i=1}^{l_\beta - 1}$ for appropriate $l_\alpha, l_\beta \in \mathbb{N}$, such that $(b_i)_{i=1}^{t-1} = \boldsymbol{\alpha} \odot \boldsymbol{\beta}$ and $l_\alpha, l_\beta \geq 2$. Let $z_\alpha = z_{\alpha,1} z_{\alpha,2} z_{\alpha,3} \ldots$ and $z_\beta = z_{\beta,1} z_{\beta,2} z_{\beta,3} \ldots$ be the fixed points of the mappings of $\Sigma^{\mathbb{N}}$, induced by the substitutions

$$\gamma \to \alpha_1 \alpha_2 \ldots \alpha_{l_\alpha - 1} \varphi(\gamma), \quad \gamma \in \Sigma,$$

and

$$\gamma \to \beta_1 \beta_2 \ldots \beta_{l_\beta - 1} \varphi(\gamma), \quad \gamma \in \Sigma, \tag{4.2}$$

respectively. Let h_α be the subword complexity function of z_α, and h_β the analogous function for z_β. For $n \in \mathbb{N}$ and $i \in \left[0, l_\beta - 1\right]$ put:

$$B_{\beta,i}^{(n)} = \left\{x_1 x_2 \ldots x_n \in \Sigma^n : \exists m \geq 0, x_j = z_{\beta, ml_\beta + i + j}, 1 \leq j \leq n\right\}.$$

Lemma 4.3. *Let $n \le l_\beta$. Then $h(n) - nh(1) = h_\beta(n) - nh_\beta(1)$.*

Proof. According to [13, Lemma 5.1]

$$\left|B_i^{(n)}\right| = \begin{cases} 1, & i \in [0, t-n-1], \\ h(1), & i \in [t-n, t-1], \end{cases}$$

$$\left|B_{\beta,i}^{(n)}\right| = \begin{cases} 1, & i \in \left[0, l_\beta - n - 1\right], \\ h_\beta(1), & i \in \left[l_\beta - n, l_\beta - 1\right]. \end{cases}$$

Let $i \in \left[0, l_\beta - n - 1\right]$. Since $n+i \le l_\beta - 1$, we have $B_i^{(n)} = \{b_{i+1}b_{i+2}\dots b_{i+n}\}$ and $B_{\beta,i}^{(n)} = \{\beta_{i+1}\beta_{i+2}\dots\beta_{i+n}\}$. Moreover, since $(b_i)_{i=1}^{t-1} = \boldsymbol{\alpha} \odot \boldsymbol{\beta}$, the definition of the compound sequence yields $B_i^{(n)} = B_{\beta,i}^{(n)}$ and $B_{i+jl_\beta}^{(n)} = B_i^{(n)} = B_{\beta,i}^{(n)}$ for each $j \in [0, l_\alpha - 1]$. Now, let $i \in \left[l_\beta - n, l_\beta - 1\right]$, and similarly to the previous case, $B_{i+jl_\beta}^{(n)} \subseteq B_{i+t-l_\beta}^{(n)}$ for each $j \in [0, l_\alpha - 1]$. Hence, we have

$$h(n) = \left|\bigcup_{j=0}^{t-1} B_j^{(n)}\right| = \left|\bigcup_{j=0}^{l_\alpha-1}\bigcup_{i=0}^{l_\beta-1} B_{i+jl_\beta}^{(n)}\right| = \left|\bigcup_{i=0}^{l_\beta-1} B_{i+t-l_\beta}^{(n)}\right|.$$

On the other hand,

$$h_\beta(n) = \left|\bigcup_{i=0}^{l_\beta-1} B_{\beta,i}^{(n)}\right|.$$

For each subset $I \subseteq \left[0, l_\beta - 1\right]$ that satisfies $|I| \ge 2$, Lemma 4.2 yields that $\left|\bigcap_{i\in I} B_{i+t-l_\beta}^{(n)}\right| \le 1$ and $\left|\bigcap_{i\in I} B_{\beta,i}^{(n)}\right| \le 1$. Moreover, since $(b_i)_{i=1}^{t-1} = \boldsymbol{\alpha} \odot \boldsymbol{\beta}$, Proposition 2.3 also yields that $\left|\bigcap_{i\in I} B_{i+t-l_\beta}^{(n)}\right| = 1$ if and only if $\left|\bigcap_{i\in I} B_{\beta,i}^{(n)}\right| = 1$. Thus, since

$$\left|B_{i+t-l_\beta}^{(n)}\right| = \left|B_{\beta,i}^{(n)}\right| = 1, \qquad i \in \left[0, l_\beta - n - 1\right],$$

and

$$\left|B_{i+t-l_\beta}^{(n)}\right| = h(1), \qquad \left|B_{\beta,i}^{(n)}\right| = h_\beta(1), \qquad i \in \left[l_\beta - n, l_\beta - 1\right],$$

it follows that

$$\left|\bigcup_{i=0}^{l_\beta-1} B_{i+t-l_\beta}^{(n)}\right| - \sum_{i=l_\beta-n}^{l_\beta-1}\left|B_{i+t-l_\beta}^{(n)}\right| = \left|\bigcup_{i=0}^{l_\beta-1} B_{\beta,i}^{(n)}\right| - \sum_{i=l_\beta-n}^{l_\beta-1}\left|B_{\beta,i}^{(n)}\right|.$$

Hence, $h(n) - nh(1) = h_\beta(n) - nh_\beta(1)$, which completes the proof. □

For $n \in \mathbb{N}$ and $i \in [0, l_\alpha - 1]$, denote:

$$B_{\alpha,i}^{(n)} = \left\{x_1x_2\dots x_n \in \Sigma^n : \exists m \ge 0, x_j = z_{\alpha,ml_\alpha+i+j}, 1 \le j \le n\right\}.$$

Lemma 4.4. *If the cyclicity of the sequence $\boldsymbol{\beta}$ is l_β, then for $n = kl_\beta + k' \in \left[l_\beta, t\right]$ with $k \ge 1$ and $k' \in \left[0, l_\beta - 1\right]$:*

$$h(n) - nh(1) = \left(l_\beta - k'\right)h_\alpha(k) + k'h_\alpha(k+1) - nh_\alpha(1).$$

Proof. Let n be as in the statement of the lemma. Since the cyclicity of $\boldsymbol{\beta}$ is l_β and $n \ge l_\beta$, by [13, Lemma 5.5] the sets $B_{\beta,i}^{(n)}$, $0 \le i \le l_\beta - 1$, are pairwise disjoint. Put:

$$C_i^{(n)} = \bigcup_{j=0}^{l_\alpha-1} B_{i+jl_\beta}^{(n)}, \quad 0 \le i \le l_\beta - 1.$$

Since $(b_i)_{i=1}^{t-1} = \boldsymbol{\alpha} \odot \boldsymbol{\beta}$ and the sets $B_{\beta,i}^{(n)}$, $0 \le i \le l_\beta - 1$, are pairwise disjoint, the sets $C_i^{(n)}$, $0 \le i \le l_\beta - 1$, are pairwise disjoint as well. Let $i' \in \left[0, l_\beta - k' - 1\right]$. For a set $J \subseteq [0, l_\alpha - 1]$ that satisfies $|J| \ge 2$, we have $\bigcap_{j\in J} B_{i'+jl_\beta}^{(n)} \le 1$ and $\bigcap_{j\in J} B_{\alpha,j}^{(k)} \le 1$. Since $(b_i)_{i=1}^{t-1} = \boldsymbol{\alpha} \odot \boldsymbol{\beta}$, Proposition 2.3 yields that $\bigcap_{j\in J} B_{i'+jl_\beta}^{(n)} = 1$ if and only if $\bigcap_{j\in J} B_{\alpha,j}^{(k)} = 1$. According to [13, Lemma 5.1],

$$\left|B_{i'+jl_\beta}^{(n)}\right| = \left|B_{\alpha,j}^{(k)}\right| = 1$$

for $j \in [0, l_\alpha - k - 1]$, and

$$\left|B_{i'+jl_\beta}^{(n)}\right| = h(1), \qquad \left|B_{\alpha,j}^{(k)}\right| = h_\alpha(1)$$

for $j \in [l_\alpha - k, l_\alpha - 1]$. Therefore,

$$\left|\bigcup_{j=0}^{l_\alpha-1} B^{(n)}_{i'+jl_\beta}\right| - \sum_{j=l_\alpha-k}^{l_\alpha-1}\left|B^{(n)}_{i'+jl_\beta}\right| = \left|\bigcup_{j=0}^{l_\alpha-1} B^{(k)}_{l_\alpha,j}\right| - \sum_{i=l_\alpha-k}^{l_\alpha-1}\left|B^{(k)}_{l_\alpha,j}\right|.$$

Thus, $\left|C^{(n)}_{i'}\right| - kh(1) = h_\alpha(k) - kh_\alpha(1)$ for $i' \in \left[0, l_\beta - k' - 1\right]$. By the same token, $\left|C^{(n)}_{i'}\right| - (k+1)h(1) = h_\alpha(k+1) - (k+1)h_\alpha(1)$ for $i' \in \left[l_\beta - k', l_\beta - 1\right]$. Since the sets $C^{(n)}_i$, $0 \le i \le l_\beta - 1$, are pairwise disjoint, we have

$$\begin{aligned}\left|\bigcup_{i=0}^{l_\beta-1} C^{(n)}_i\right| - nh(1) &= \left|\bigcup_{i=0}^{l_\beta-k'-1} C^{(n)}_i\right| - \left(l_\beta - k'\right)kh(1) + \left|\bigcup_{i=l_\beta-k'}^{l_\beta-1} C^{(n)}_i\right| - k'(k+1)h(1)\\ &= \left(l_\beta - k'\right)h_\alpha(k) - \left(l_\beta - k'\right)kh_\alpha(1) + k'h_\alpha(k+1) - k'(k+1)h_\alpha(1).\end{aligned}$$

Thus,

$$h(n) - nh(1) = \left(l_\beta - k'\right)h_\alpha(k) + k'h_\alpha(k+1) - nh_\alpha(1),$$

which completes the proof. □

4.3. Proof of Proposition 3.2

The proposition actually states that $\underline{L} < h(1)$ if and only if there exist two sequences $\boldsymbol{\alpha} = (\alpha_i)_{i=1}^{l_\alpha-1}$ and $\boldsymbol{\beta}$ such that $(b_i)_{i=1}^{t-1} = \boldsymbol{\alpha} \odot \boldsymbol{\beta}$ and the sequence $\boldsymbol{\alpha}$ satisfies the stated conditions.

The following lemma presents conditions which imply $\liminf_{n\to\infty} h(n)/n < h(1)$.

Lemma 4.5. *We have $h(2) = 2h(1) - 1$ if and only if the sequence $(b_i)_{i=1}^{t}$ satisfies the following conditions:*

(1) $b_1 = b_{t-1}$,
(2) *for each $i \in [1, t-2]$, either $b_i = b_1$ or $b_{i+1} = b_1$.*

Proof. According to [13, Lemma 5.1] we have $\left|B^{(2)}_{t-1}\right| = \left|B^{(2)}_{t-2}\right| = h(1)$, and since $b_{t-1}b_1 \in B^{(2)}_{t-1}, B^{(2)}_{t-2}$ we have $\left|B^{(2)}_{t-1} \cup B^{(2)}_{t-2}\right| \le 2h(1) - 1$. On the other hand, $B^{(2)}_{t-1} \cap B^{(2)}_{t-2} \subseteq \{b_{t-1}b_1\}$, and hence $\left|B^{(2)}_{t-1} \cup B^{(2)}_{t-2}\right| = 2h(1) - 1$. First, suppose that the sequence $(b_i)_{i=1}^{t}$ satisfies the conditions of the lemma. Since $B^{(2)}_i = \{b_{i+1}b_{i+2}\}$ for each $i \in [0, t-3]$, either $b_{i+1} = b_1 = b_{t-1}$ or $b_{i+2} = b_1$, and hence either $B^{(2)}_i \subseteq B^{(2)}_{t-2}$ or $B^{(2)}_i \subseteq B^{(2)}_{t-1}$. Thus, $B^{(2)}_i \subseteq B^{(2)}_{t-1} \cup B^{(2)}_{t-2}$ for each $i \in [0, t-1]$, and therefore

$$h(2) = \left|\bigcup_{i=0}^{t-1} B^{(2)}_i\right| = \left|B^{(2)}_{t-1} \cup B^{(2)}_{t-2}\right| = 2h(1) - 1.$$

On the other hand, suppose now that $h(2) = 2h(1) - 1$. Since $\left|B^{(2)}_{t-1} \cup B^{(2)}_{t-2}\right| = 2h(1) - 1$, it follows that $B^{(2)}_i \subseteq B^{(2)}_{t-1} \cup B^{(2)}_{t-2}$ for each $i \in [0, t-3]$. Since $B^{(2)}_i = \{b_{i+1}b_{i+2}\}$ for $i \in [0, t-3]$, it means that for each $i \in [0, t-3]$ either $b_{i+1} = b_{t-1}$ or $b_{i+2} = b_1$. Suppose that $b_{t-1} \neq b_1$, and let $j \in [1, t-1]$ be the smallest index for which $b_j \neq b_1$ (since $b_{t-1} \neq b_1$ the index j is well defined). Obviously $j > 1$, and hence $j - 2 \in [0, t-3]$. Therefore, either $b_{j-1} = b_{t-1}$ or $b_j = b_1$. Now, the definition of j yields that $b_{j-1} = b_1 \neq b_{t-1}$, while on the other hand $b_j \neq b_1$. Thus, we have a contradiction, and therefore $b_1 = b_{t-1}$. Moreover, $i - 1 \in [0, t-3]$ for each $i \in [1, t-2]$, and hence either $b_i = b_{t-1} = b_1$ or $b_{i+1} = b_1$, which completes the proof. □

Lemma 4.6. *Let $m \in [1, t-1]$, and suppose that $h(m+1) - h(m) = h(1) - 1$. Then $B^{(m+1)}_j \cap B^{(m+1)}_{j'} \neq \emptyset$ for $j, j' \in [0, t-1]$ if and only if $B^{(m)}_j \cap B^{(m)}_{j'} \neq \emptyset$.*

Proof. Select two indices $j, j' \in [0, t-1]$. Obviously, if $B^{(m+1)}_j \cap B^{(m+1)}_{j'} \neq \emptyset$ then $B^{(m)}_j \cap B^{(m)}_{j'} \neq \emptyset$ as well. On the other hand, Lemma 4.2 yields that, for any set $I \subseteq [0, t-1]$ with $|I| \ge 2$, we have $\left|\bigcup_{i\in I} B^{(n)}_i\right| \le 1$ for both $n = m$ and $n = m+1$. For each $n \in \mathbb{N}$:

$$h(n) = \left|\bigcup_{i=0}^{t-1} B^{(n)}_i\right| = \sum_{i=0}^{t-1}\left|B^{(n)}_i\right| - \sum_{i_1,i_2\in[0,t-1]}\left|B^{(n)}_{i_1} \cap B^{(n)}_{i_2}\right| + \cdots.$$

Since $\sum_{i=0}^{t-1}\left|B^{(m+1)}_i\right| - \sum_{i=0}^{t-1}\left|B^{(m)}_i\right| = h(1) - 1$, which is the same as $h(m+1) - h(m)$, and for any subset $I \subseteq [0, t-1]$ we have $\left|\bigcup_{i\in I} B^{(m+1)}_i\right| = 1$ only if $\left|\bigcup_{i\in I} B^{(m)}_i\right| = 1$, it follows that $B^{(m)}_j \cap B^{(m)}_{j'} \neq \emptyset$ yields $B^{(m+1)}_j \cap B^{(m+1)}_{j'} \neq \emptyset$, as otherwise we would have $h(m+1) - h(m) > h(1) - 1$. □

The following lemma is proved similarly.

Lemma 4.7. *Let $m \in [1, t-1]$, and suppose that $h(m+1) - h(m) = h(1) - 1$. Then $B_j^{(m+1)} \cap B_{j'}^{(m+1)} \neq \emptyset$ for $j, j' \in [0, t-1]$ if and only if $B_{(j+1) \bmod t}^{(m)} \cap B_{(j'+1) \bmod t}^{(m)} \neq \emptyset$.*

The lemmas above yield the following corollary.

Corollary 4.8. *Let $m \in [1, t-1]$, and suppose that $h(m+1) - h(m) = h(1) - 1$. Then for $j, j' \in [0, t-1]$ we have $B_j^{(m)} \cap B_{j'}^{(m)} \neq \emptyset$ if and only if $B_{(j+1) \bmod t}^{(m)} \cap B_{(j'+1) \bmod t}^{(m)} \neq \emptyset$.*

The corollary helps us proving the following lemma.

Lemma 4.9. *Let $m \in [1, t-1]$. If $h(m) = mh(1)$ and $h(m+1) = (m+1)h(1) - 1$, then the sets $B_i^{(m)}$, $i \in [t-m, t-1]$, are pairwise disjoint and*

$$\bigcup_{i=0}^{t-m-1} B_i^{(m)} \subseteq \bigcup_{i=t-m}^{t-1} B_i^{(m)}. \tag{4.3}$$

Proof. According to [13, Lemma 5.1], we have $\left|B_i^{(m)}\right| = 1$ for each $i \in [0, t-m-1]$. Out of each collection of equal $B_i^{(m)}$'s not contained in $\bigcup_{i \in [t-m, t-1]} B_i^{(m)}$, take the index of that $B_i^{(m)}$ with the smallest possible i, to from jointly with the set $[t-m, t-1]$ a set $I \subseteq [0, t-1]$. Let j be the smallest element of I. We claim that $j = t-m$. Suppose that $j < t-m$. According to [13, Lemma 5.1], this implies that $\left|B_j^{(m)}\right| = 1$ and $\left|B_0^{(m)}\right| = 1$. First, in case $j > 0$, the definition of I yields that $B_0^{(m)} \subseteq \bigcup_{i \in [t-m, t-1]} B_i^{(m)}$ and $B_j^{(m)} \cap B_0^{(m)} = \emptyset$. The definition of I also yields that $B_j^{(m)} \cap B_i^{(m)} = \emptyset$ for each $i \in [t-m+1, t-1]$, and hence Corollary 4.8 yields that $B_{j-1}^{(m)} \cap B_i^{(m)} = \emptyset$ for each $i \in [t-m, t-1]$. Thus, by the definition of I we have $j-1 \in I$, which contradicts the definition of j.

Now suppose that $j = 0$, and let $k \in [1, m]$ be the maximal value for which the sets $B_i^{(m)}$, $i \in [t-k, t-1]$, are pairwise disjoint. The definition of I and j implies that $B_0^{(m)} \cap B_i^{(m)} = \emptyset$ for $i \in [t-m, t-1]$. Hence, in case $k < m$, Corollary 4.8 yields that the sets $B_i^{(m)}$, $i \in [t-k-1, t-1]$, are pairwise disjoint as well, which contradicts the definition of k. Therefore $k = m$, and since [13, Lemma 5.1] implies that $\left|B_{t-i}^{(m)}\right| = h(1)$ for $i \in [t-m, t-1]$, it turns out that

$$h(m) = \left|\bigcup_{i=0}^{t-1} B_i^{(m)}\right| \geq mh(1) + 1,$$

which is a contradiction. Thus, $j = t-m$, which yields (4.3). Since [13, Lemma 5.1] implies that $\left|B_i^{(m)}\right| = h(1)$ for $i \in [t-m, t-1]$, and $h(m) = mh(1)$, it also implies that the sets $B_i^{(m)}$, $t-m \leq i \leq t-1$, are pairwise disjoint. This completes the proof. □

Lemma 4.10. *Let $m \in [1, t-1]$. If*

$$h(m) = mh(1), \qquad h(m+1) = (m+1)h(1) - 1,$$

then for an appropriate $l_\alpha \geq 2$ there exist two sequences $\boldsymbol{\alpha} = (\alpha_i)_{i=1}^{l_\alpha - 1}$ and $\boldsymbol{\beta} = (\beta_i)_{i=1}^{l_\beta - 1}$, for $l_\beta = m$, such that the sequence $(b_i)_{i=1}^{t-1} = \boldsymbol{\alpha} \odot \boldsymbol{\beta}$, and the cyclicity of $\boldsymbol{\beta}$ is l_β.

Proof. In case $m = 1$, the statement is obviously true for $l_\alpha = t$, so suppose that $m > 1$. Lemma 4.9 yields that the sets $B_i^{(m)}$, $t-m \leq i \leq t-1$, are pairwise disjoint and

$$\bigcup_{i=0}^{t-m-1} B_i^{(m)} \subseteq \bigcup_{i=t-m}^{t-1} B_i^{(m)}.$$

Therefore, $B_0^{(m)} \cap B_j^{(m)} \neq \emptyset$ for some $j \in [t-m, t-1]$. Corollary 4.8 implies that $B_{t-1}^{(m)} \cap B_{j-1}^{(m)} \neq \emptyset$, and since the sets $B_i^{(m)}$, $t-m \leq i \leq t-1$, are pairwise disjoint it follows that $j-1 \neq [t-m, t-1]$. Hence $j = t-m$, which means that $B_0^{(m)} \cap B_{t-m}^{(m)} \neq \emptyset$. Applying Corollary 4.8 repeatedly, we conclude that $B_{m-1}^{(m)} \cap B_{t-1}^{(m)} \neq \emptyset$, and hence $B_m^{(m)} \cap B_0^{(m)} \neq \emptyset$. Since $m < t$, [13, Lemma 5.1] implies that $\left|B_0^{(m)}\right| = 1$, and therefore $B_m^{(m)} \cap B_{t-m}^{(m)} \neq \emptyset$. Thus, by Corollary 4.8, $B_i^{(m)} \cap B_{i \bmod m+t-m}^{(m)} \neq \emptyset$ for each $i \in [0, t-1]$. In particular, for $j = t-1 \bmod m + t - m$ we obtain $B_{t-1}^{(m)} \cap B_j^{(m)} \neq \emptyset$. Since $j \in [t-m, t-1]$, and the sets $B_i^{(m)}$, $t-m \leq i \leq t-1$, are pairwise disjoint, it follows that $j = t-1$. Therefore $t-1 \bmod m = m-1$, and hence $m|t$.

Put $l_\alpha = t/m$, and let $k \in [0, l_\alpha - 1]$. We have $B_{k\cdot m}^{(m)} \cap B_{t-m}^{(m)} \neq \emptyset$, and hence Proposition 2.3 and the definitions of $B_{k\cdot m}^{(m)}$ and $B_{t-m}^{(m)}$ yield that $b_{k\cdot m+i} = b_{t-m+i}$ for each $i \in [1, m-1]$. Put $\beta_i = b_{t-m+i}$ for $i \in [1, m-1]$, and $\alpha_i = b_{i\cdot m}$ for $i \in [1, l_\alpha - 1]$. Since $b_{k\cdot m+i} = b_{t-m+i} = \beta_i$ for each $k \in [0, l_\alpha - 1]$ and $i \in [1, m-1]$, it turns out that $(b_i)_{i=1}^{t-1} = \boldsymbol{\alpha} \odot \boldsymbol{\beta}$ for $\boldsymbol{\alpha} = (\alpha_i)_{i=1}^{l_\alpha - 1}$ and $\boldsymbol{\beta} = (\beta_i)_{i=1}^{l_\beta - 1}$, where $l_\beta = m$. Let $z_\beta = z_{\beta,1} z_{\beta,2} z_{\beta,3} \ldots$ be the fixed point of the mapping of $\Sigma^{\mathbb{N}}$, induced by the substitution in (4.2), and let h_β be the subword complexity function of z_β. Since $h\left(l_\beta\right) = l_\beta h(1)$, Lemma 4.3 shows that $h_\beta\left(l_\beta\right) = l_\beta h_\beta(1)$. Therefore, according to [13, Lemma 5.11], the cyclicity of $\boldsymbol{\beta}$ is l_β. This completes the proof. □

We are now ready to prove Proposition 3.2.

Proof of Proposition 3.2. First, in case $c < t$ we have $h(1) - (t-c)/t \in A$ and the sequence $(b_i)_{i=1}^{t-1}$ is c-periodic. Therefore $\underline{L} < h(1)$, and on the other hand the fact that $c|t$ yields that $(b_i)_{i=1}^{t-1} = \boldsymbol{\alpha} \odot \boldsymbol{\beta}$, where $\boldsymbol{\alpha} = (\alpha_i)_{i=1}^{t/c-1} = (b_{ic})_{i=1}^{t/c-1}$ and $\boldsymbol{\beta} = (b_i)_{i=1}^{c-1}$. Since $c < t$ and $c|t$ we have $t/c \geq 2$, and since $(b_i)_{i=1}^{t-1}$ is c-periodic we have $\alpha_i = b_{ic} = b_c$ for each $i \in [1, t/c-1]$. Thus, $\alpha_1 = \alpha_{t/c-1}$, and $\alpha_i = \alpha_1$ for each $i \in [1, t/c-2]$, which completes the proof for the case $c < t$.

Now, suppose that $c = t$. Then $N = t$ and $\underline{L} = \min_{1 \leq m \leq t} h(m)/m$. First, suppose that there exist two sequences $\boldsymbol{\alpha} = (\alpha_i)_{i=1}^{l_\alpha-1}$ and $\boldsymbol{\beta} = (\beta_i)_{i=1}^{l_\beta-1}$, such that $(b_i)_{i=1}^{t-1} = \boldsymbol{\alpha} \odot \boldsymbol{\beta}$, and $\boldsymbol{\alpha}$ satisfies the conditions of the proposition. Let $z_\alpha = z_{\alpha,1} z_{\alpha,2} z_{\alpha,3} \ldots$ and $z_\beta = z_{\beta,1} z_{\beta,2} z_{\beta,3} \ldots$ be the fixed points of the mappings of $\Sigma^{\mathbb{N}}$, induced by the substitutions

$$\gamma \to \alpha_1 \alpha_2 \ldots \alpha_{l_\alpha-1} \varphi(\gamma), \quad \gamma \in \Sigma,$$

and

$$\gamma \to \beta_1 \beta_2 \ldots \beta_{l_\beta-1} \varphi(\gamma), \quad \gamma \in \Sigma,$$

respectively. Let h_α be the subword complexity function of z_α, and h_β the analogous function for z_β. In case the cyclicity of $\boldsymbol{\beta}$ is less than l_β, the sequence $\boldsymbol{\beta}$ is not the empty sequence, and [13, Lemma 5.11] yields that $h_\beta\left(m_\beta\right) = m_\beta h_\beta(1) - m_\beta + c_\beta$, where c_β is the cyclicity of $\boldsymbol{\beta}$. Hence, if the cyclicity of $\boldsymbol{\beta}$ is less than l_β, then Lemma 4.3 yields that $h\left(l_\beta\right) = l_\beta h(1) - l_\beta + c_2$, and since $c_\beta < l_\beta$ we have $h\left(l_\beta\right) < l_\beta h(1)$, which straightforwardly means that $\underline{L} < h(1)$. Now, suppose that the cyclicity of $\boldsymbol{\beta}$ is l_β. Since $\boldsymbol{\alpha}$ satisfies the conditions of the proposition, Lemma 4.5 yields that $h_\alpha(2) = 2h_\alpha(1) - 1$. Thus, Lemma 4.4 implies that

$$h\left(2l_\beta\right) - 2l_\beta h(1) = l_\beta h_\alpha(2) - 2l_\beta h(1) = -l_\beta,$$

and hence $h\left(2l_\beta\right)/2l_\beta < h(1)$. Therefore, $\underline{L} < h(1)$.

Next, suppose that $\underline{L} < h(1)$, so that $h(n) < nh(1)$ for some $n \in [2, t]$. According to [13, Lemma 6.2] we have $h(i+1) - h(i) \geq h(1) - 1$ for each $i \in [1, t-1]$, and therefore there exists an $m \in [1, t-1]$ such that $h(m) = mh(1)$ and $h(m+1) = (m+1)h(1) - 1$. Thus, Lemma 4.10 proves the existence of an integer $l_\alpha \geq 2$ and two sequences $\boldsymbol{\alpha} = (\alpha_i)_{i=1}^{l_\alpha-1}$ and $\boldsymbol{\beta} = (\beta_i)_{i=1}^{l_\beta-1}$, for $l_\beta = m$, such that $(b_i)_{i=1}^{t-1} = \boldsymbol{\alpha} \odot \boldsymbol{\beta}$, and the cyclicity of $\boldsymbol{\beta}$ is l_β. Since the cyclicity of $\boldsymbol{\beta}$ is l_β, Lemma 4.4 gives

$$h\left(l_\beta + 1\right) - \left(l_\beta + 1\right) h(1) = \left(l_\beta - 1\right) h_\alpha(1) + h_\alpha(2) - \left(l_\beta + 1\right) h_\alpha(1).$$

This means that $h_\alpha(2) - 2h_\alpha(1) = -1$. Therefore, Lemma 4.5 yields that $\alpha_1 = \alpha_{l_\alpha-1}$, and for each $i \in [1, l_\alpha - 2]$ either $\alpha_i = \alpha_1$ or $\alpha_{i+1} = \alpha_1$. This completes the proof. □

4.4. Necessity of the conditions in Theorem 3.5

Lemma 4.11. *Suppose that $t > 2$ and $h(t-1) = (t-1)h(1)$. Then there exist two indexes, $m, m' \in [1, t-1]$, such that $m > m'$, $B_m^{(t-1)} \cap B_{m'}^{(t-1)} \neq \emptyset$, and $B_i^{(t-1)} \cap B_j^{(t-1)} = \emptyset$ for each pair $(i, j) \in [0, t-1]^2$ which satisfies $i > j$ and $(i, j) \neq \left(m, m'\right)$.*

Proof. According to [13, Lemma 5.1] we have $\left|B_0^{(t-1)}\right| = 1$ and $\left|B_i^{(t-1)}\right| = h(1)$ for each $i \in [1, t-1]$. Since $\left|\bigcup_{i=0}^{t-1} B_i^{(t-1)}\right| = h(t-1) = (t-1)h(1)$, it turns out that the sets $B_i^{(t-1)}$, $i \in [0, t-1]$, are not pairwise disjoint, but there is a single pair $\left(m, m'\right) \in [0, t-1]^2$ such that $m > m'$ and $B_m^{(t-1)} \cap B_{m'}^{(t-1)} \neq \emptyset$. In case $m' = 0$, Proposition 2.3 and the definition of the sets $B_m^{(t-1)}$ and $B_{m'}^{(t-1)}$ yield that $b_i = b_{i+m \bmod t}$ for each $i \in [0, t-1] \setminus \{t-m\}$. Therefore $B_{t-1}^{(t-1)} \cap B_{m-1}^{(t-1)} \neq \emptyset$, and since $t > 2$ we have either $t - 1 \neq m$ or $m - 1 \neq 0$. Thus $\left|\bigcup_{i=0}^{t-1} B_i^{(t-1)}\right| \leq h(t-1) - 1$, which is a contradiction. Hence $m' > 0$, which completes the proof. □

Lemma 4.12. *Suppose that there exist two indices $m, m' \in [1, t-1]$, such that $m > m'$, $B_m^{(t-1)} \cap B_{m'}^{(t-1)} \neq \emptyset$, and $B_i^{(t-1)} \cap B_j^{(t-1)} = \emptyset$ for each pair $(i, j) \in [0, t-1]^2$ which satisfies $i > j$ and $(i, j) \neq \left(m, m'\right)$. Then there exist two sequences $\boldsymbol{\alpha} = (\alpha_i)_{i=1}^{l_\alpha-1}$ and $\boldsymbol{\beta} = (\beta_i)_{i=1}^{l_\beta-1}$ such that $(b_i)_{i=1}^{t-1} = \boldsymbol{\alpha} \odot \boldsymbol{\beta}$, and there exist $k, k' \in [1, l_\alpha - 1]$ such that $\gcd(k, l_\alpha) = 1$ and*

$$\alpha_i = \alpha_{i+k \bmod l_\alpha}, \quad i \in [1, l_\alpha - 1] \setminus \{l_\alpha - k, k'\},$$
$$\alpha_{k'} \neq \alpha_{k'+k \bmod l_\alpha}.$$

Proof. Since $B_m^{(t-1)} \cap B_{m'}^{(t-1)} \neq \emptyset$, Proposition 2.3 and the definition of the sets $B_m^{(t-1)}$ and $B_{m'}^{(t-1)}$ give $b_i = b_{i+(m-m') \bmod t}$ for each $i \in [1, t-1] \setminus \{m', t-(m-m')\}$. On the other hand, since $m, m' \geq 1$ and $B_{m-1}^{(t-1)} \cap B_{m'-1}^{(t-1)} = \emptyset$, we have $b_{m'} \neq b_m$.

Since $b_i = b_{i+(m-m') \bmod t}$ for each $i \in [1, t-1] \setminus \{m', t-(m-m')\}$, for $l_\beta = \gcd(m-m', t)$ we have $b_i = b_{i+l_\beta \bmod t}$ for each $i \in [1, t-1]$ satisfying $l_\beta \nmid i$. It follows that $l_\beta | m'$. Moreover, $(b_i)_{i=1}^{t-1} = \boldsymbol{\alpha} \odot \boldsymbol{\beta}$ for $\boldsymbol{\beta} = (b_i)_{i=1}^{l_\beta - 1}$ and $\boldsymbol{\alpha} = (\alpha_i)_{i=1}^{l_\alpha - 1} = \left(b_{i \cdot l_\beta}\right)_{i=1}^{l_\alpha - 1}$, where $l_\alpha = t/l_\beta$. On the other hand, for $k = (m-m')/l_\beta$ and $k' = m'/l_\beta$:

$$\alpha_i = b_{i \cdot l_\beta} = b_{i \cdot l_\beta + m - m' \bmod t} = \alpha_{i+k \bmod l_\alpha}, \qquad i \in [1, l_\alpha] \setminus \{l_\alpha - k, k'\},$$
$$\alpha_{k'} = b_{m'} \neq b_{m' + m - m' \bmod t} = \alpha_{k'+k \bmod l_\alpha}.$$

This completes the proof. □

Now, we are ready to prove that the conditions in Theorem 3.5 are necessary for $\underline{L} = \bar{L} > 0$.

Lemma 4.13. *If $\underline{L} = \bar{L} > 0$, then there exist two sequences $\boldsymbol{\alpha} = (\alpha_i)_{i=1}^{l_\alpha - 1}$ and $\boldsymbol{\beta} = (\beta_i)_{i=1}^{l_\beta - 1}$, satisfying the conditions in Theorem 3.5, such that $(b_i)_{i=1}^{t-1} = \boldsymbol{\alpha} \odot \boldsymbol{\beta}$.*

Proof. Suppose that $\underline{L} = \bar{L} > 0$. As was observed in the course of the proof of Proposition 3.2, the condition $\underline{L} = \bar{L} > 0$ implies that $c = t$ and $h(n)/n = h(1)$ for each $n \in [1, t-1]$. In particular, $h(t-1) = (t-1)h(1)$, and therefore Lemmas 4.11 and 4.12 yield the existence of two sequences $\boldsymbol{\alpha} = (\alpha_i)_{i=1}^{l_\alpha - 1}$ and $\boldsymbol{\beta} = (\beta_i)_{i=1}^{l_\beta - 1}$, such that $(b_i)_{i=1}^{t-1} = \boldsymbol{\alpha} \odot \boldsymbol{\beta}$, and there exist $k, k' \in [1, l_\alpha - 1]$ such that $\gcd(k, l_\alpha) = 1$ and

$$\alpha_i = \alpha_{i+k \bmod l_\alpha}, \quad i \in [1, l_\alpha - 1] \setminus \{l_\alpha - k, k'\},$$
$$\alpha_{k'} \neq \alpha_{k'+k \bmod l_\alpha}.$$

Moreover, the equality $\underline{L} = h(1)$ and Proposition 3.2 show that $l_\alpha \geq 3$.

Now, suppose that $\alpha_1 = \alpha_{l_\alpha - 1}$. The conditions on the sequence $\boldsymbol{\alpha}$ imply that $|\{\alpha_i : 1 \leq i \leq l_\alpha - 1\}| = 2$. Moreover, $\alpha_k \neq \alpha_{l_\alpha - k}$, as otherwise the equality $\alpha_i = \alpha_{i+k \bmod l_\alpha}$ for $i \in [1, l_\alpha - 1] \setminus \{l_\alpha - k, k'\}$ would give $|\{\alpha_i : 1 \leq i \leq l_\alpha - 1\}| = 1$. Without loss of generality, we may suppose that $\alpha_1 = \alpha_{l_\alpha - k}$, and $\alpha_{l_\alpha - 1} = \alpha_{l_\alpha - k}$. Let $r \in [1, l_\alpha - 1]$ be such that $r \cdot k \bmod l_\alpha = k'$. The conditions on the sequence $\boldsymbol{\alpha}$ imply that, if for a given $i \in [1, l_\alpha - 1]$ we have $i = j \cdot k \bmod l_\alpha$ for some $j \in [1, r]$, then $\alpha_i = \alpha_k$. Since $\underline{L} = h(1)$, Proposition 3.2 yields that there exists some $i \in [1, l_\alpha - 2]$ such that $\alpha_i \neq \alpha_1$ and $\alpha_{i+1} \neq \alpha_1$. In other words, $\alpha_i = \alpha_{i+1} = \alpha_k$. Let $j, j' \in [1, l_\alpha]$ be such that $i = j \cdot k \bmod l_\alpha$ and $i + 1 = j' \cdot k \bmod l_\alpha$. Since $\alpha_i = \alpha_{i+1} = \alpha_k$, we have $j, j' \in [1, r]$. If $j' > j$, then $1 = (j' - j) \cdot k \bmod l_\alpha$, and since $j' - j \in [1, r]$ we obtain $\alpha_1 = \alpha_k$, which is a contradiction. Thus, $j' < j$. Therefore, $l_\alpha - 1 = (j - j') \cdot k \bmod l_\alpha$, and since $j - j' \in [1, r]$ we obtain $\alpha_{l_\alpha - 1} = \alpha_k$, which is a contradiction as well. Thus, we have a contradiction, and therefore $\alpha_1 \neq \alpha_{l_\alpha - 1}$. Hence, the sequence $\boldsymbol{\alpha}$ is appropriate.

Finally, suppose that $l_\beta > 1$, and let h_β be the subword complexity function of the D0L word formed by the substitution

$$\gamma \to \beta_1 \beta_2 \ldots \beta_{l_\beta - 1} \varphi(\gamma), \quad \gamma \in \Sigma. \tag{4.4}$$

Since $h(n) = nh(1)$ for each $n \in [1, t-1]$, and in particular for $n \in [1, l_\beta]$, Lemma 4.3 implies that $h_\beta(n) = nh_\beta(1)$ for each $n \in [1, l_\beta]$. In particular, we have $h_\beta(l_\beta) = l_\beta h_\beta(1)$, and therefore [13, Lemma 5.11] yields that the cyclicity of $\boldsymbol{\beta}$ is l_β. Therefore, according to [13, Theorem 3.3], we have

$$\limsup_{n\to\infty} \frac{h_\beta(n)}{n} = \max_{1 \leq n \leq l_\beta - 1} \frac{h_\beta(n)}{n} = h_\beta(1),$$
$$\liminf_{n\to\infty} \frac{h_\beta(n)}{n} = \min_{1 \leq n \leq m_{\beta - 1}} \frac{h_\beta(n)}{n} = h_\beta(1).$$

Thus,

$$\liminf_{n\to\infty} \frac{h_\beta(n)}{n} = \limsup_{n\to\infty} \frac{h_\beta(n)}{n} = h_\beta(1),$$

and hence the substitution defined by (4.4) is steady. Thus, the sequences $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfy the conditions in Theorem 3.5, which completes the proof. □

4.5. *Sufficiency of the conditions in Theorem 3.5*

In this subsection we suppose that the sequence $(b_i)_{i=1}^{t-1}$ satisfies the conditions in Theorem 3.5, i.e., there exist two sequences $\boldsymbol{\alpha} = (\alpha_i)_{i=1}^{l_\alpha - 1}$ and $\boldsymbol{\beta} = (\beta_i)_{i=1}^{l_\beta - 1}$, satisfying the conditions of the theorem, such that $(b_i)_{i=1}^{t-1} = \boldsymbol{\alpha} \odot \boldsymbol{\beta}$.

Let $z_\alpha = z_{\alpha,1} z_{\alpha,2} z_{\alpha,3} \ldots$ be the fixed point of the mapping of $\Sigma^{\mathbb{N}}$, induced by the substitution

$$\gamma \to \alpha_1 \alpha_2 \ldots \alpha_{l_\alpha - 1} \varphi(\gamma), \quad \gamma \in \Sigma.$$

Let h_α be the subword complexity function of z_α, and define the sets

$$B_{\alpha,i}^{(n)} = \left\{x_1x_2\dots x_n \in \Sigma^n : \exists m \geq 0, x_j = z_{\alpha,mt+i+j}\,\forall j\,(1 \leq j \leq n)\right\}$$

for $n \in \mathbb{N}$ and $i \in [0, l_\alpha - 1]$.

Lemma 4.14. $h_\alpha(n) \leq nh_\alpha(1)$ *for each* $n \in [1, l_\alpha]$.

Proof. The inequality is obvious for $n = 1$, so let $n \in [2, l_\alpha]$. According to [13, Lemma 5.1] we have $\left|B_{\alpha,i}^{(n)}\right| = 1$ for $i \in [0, l_\alpha - n - 1]$, and $\left|B_{\alpha,i}^{(n)}\right| = h(1)$ for $i \in [l_\alpha - n, l_\alpha - 1]$. Since $\alpha_i = \alpha_{i+k \bmod l_\alpha}$ for each $i \in [1, l_\alpha - 1] \setminus \{k', l_\alpha - k\}$, if for a given $i \in [0, l_\alpha - 1]$ we have $i + j \bmod l_\alpha \neq k'$ for each $j \in [1, n]$, then $B_{\alpha,i}^{(n)} \cap B_{\alpha,i+k \bmod l_\alpha}^{(n)} \neq \emptyset$. Therefore, putting

$$I = \begin{cases} [0, l_\alpha - 1] \setminus [k' - n, k'], & k' - n \geq 0, \\ [k' + 1, l_\alpha - 1] \cup [k' - n \bmod l_\alpha, l_\alpha - 1], & k' - n < 0, \end{cases}$$

we have $B_{\alpha,i}^{(n)} \cap B_{\alpha,i+k \bmod l_\alpha}^{(n)} \neq \emptyset$ for each $i \in I$. Thus,

$$\begin{aligned} \left|\bigcup_{i=0}^{l_\alpha - 1} B_{\alpha,i}^{(n)}\right| &\leq \sum_{i=0}^{l_\alpha - 1} \left|B_{\alpha,i}^{(n)}\right| - \sum_{i \in I} \left|B_{\alpha,i} \cap B_{\alpha,i+k \bmod l_\alpha}\right| \\ &\leq nh_\alpha(1) + l_\alpha - n - (l_\alpha - n) \\ &= nh_\alpha(1), \end{aligned}$$

and therefore $h_\alpha(n) \leq nh_\alpha(1)$, which completes the proof. □

From now on, let $m \in [1, l_\alpha - 1]$ be such that $m \cdot k \equiv k' \pmod{l_\alpha}$. Note that, since $l_\alpha \nmid k'$ and $\gcd(k, l_\alpha) = 1$, the integer m is well defined.

Lemma 4.15. *Let* $j \in [1, l_\alpha - 1]$*, and put* $i = j \cdot k \bmod l_\alpha$*. If* $j \in [1, m]$ *then* $\alpha_i = \alpha_k$*, while if* $j \in [m + 1, l_\alpha - 1]$ *then* $\alpha_i = \alpha_{l_\alpha - k}$.

Proof. Let $j' \in [1, l_\alpha - 1]$ be the smallest index for which $\alpha_{j' \cdot k \bmod l_\alpha} \neq \alpha_k$. Obviously, $j' > 1$, and hence $\alpha_{(j'-1)k \bmod l_\alpha} \neq \alpha_{j' \cdot k \bmod l_\alpha}$. Since $\boldsymbol{\alpha}$ is an appropriate sequence, it follows that $(j' - 1)\,k \bmod l_\alpha = k'$, and hence $j' - 1 = m$. Thus, if $j \in [1, m]$ then $\alpha_i = \alpha_k$.

On the other hand, let $j' \in [1, l_\alpha - 1]$ be the maximal index for which $\alpha_{j' \cdot k \bmod l_\alpha} \neq \alpha_{l_\alpha - k}$. Obviously, $j' < l_\alpha - 1$, and hence $\alpha_{j' \cdot k \bmod l_\alpha} \neq \alpha_{(j'+1)k \bmod l_\alpha}$. Since $\boldsymbol{\alpha}$ is an appropriate sequence, it follows that $j' \cdot k \bmod l_\alpha = k'$, and hence $j' = m$. Thus, if $j \in [m + 1, l_\alpha - 1]$ then $\alpha_i = \alpha_{l_\alpha - k}$. □

Since $\boldsymbol{\alpha}$ is an appropriate sequence, we also have $\alpha_1 \neq \alpha_{l_\alpha - 1}$, and hence the lemma above yields the following corollary.

Corollary 4.16. *We have* $\alpha_k \neq \alpha_{l_\alpha - k}$.

Lemma 4.17. $h_\alpha(n) \geq nh_\alpha(1)$ *for each* $n \in [1, l_\alpha]$.

Proof. Suppose that there exists some $n \in [1, l_\alpha]$ such that $h_\alpha(n) < nh_\alpha(1)$. According to [13, Theorem 3.3], this implies

$$\liminf_{n \to \infty} \frac{h_\alpha(n)}{n} < h_\alpha(1).$$

Hence Proposition 3.2 yields the existence of an $r | l_\alpha$, such that $r < l_\alpha$, $\alpha_r = \alpha_{l_\alpha - r}$, and:

- $\alpha_{i+r} = \alpha_i$ for each $i \in [1, l_\alpha - r - 1]$ with $r \nmid i$.
- either $\alpha_i = \alpha_r$ or $\alpha_{i+r} = \alpha_r$ for each $i \in [1, l_\alpha - r - 1]$ with $r | i$.

Since $\alpha_1 \neq \alpha_{l_\alpha - 1}$, we have $r > 1$. On the other hand, in case $k = 1$ we have $\alpha_i = \alpha_1$ for $i \in [1, k']$, and $\alpha_i = \alpha_{l_\alpha - 1}$ for $i \in [k' + 1, l_\alpha - 1]$. Therefore, in case $k' \leq r$ we have $\alpha_1 \neq \alpha_{r+1}$ which is a contradiction, and in case $k' > r$ we have $\alpha_{r-1} \neq \alpha_{m_\alpha - 1}$ which is also a contradiction. Thus, $k > 1$.

Since $\alpha_1 \neq \alpha_{m-1}$ and $k > 1$, Lemma 4.15 yields that $m \in [2, l_\alpha - 3]$. Let $r' \in [1, l_\alpha - 1]$ be such that $r' \cdot k \equiv r \pmod{k}$. Since $r | l_\alpha$, $r > 1$, and $\gcd(k, l_\alpha) = 1$, we have $r' \in [2, l_\alpha - 2]$. Suppose that $r' = l_\alpha - 2$. Let $i = (m + 2)\,k \bmod l_\alpha$. Since $r' = l_\alpha - 2$, we have $i' = m \cdot k \bmod l_\alpha$ for $i' = i + r \bmod l_\alpha$. Lemma 4.15 and Corollary 4.16 yield that $\alpha_i \neq \alpha_{i'}$, and therefore $r | i, i'$. The definitions of i and i' give $j = (m + 1)\,k \bmod l_\alpha$ and $j' = (m - 1)\,k \bmod l_\alpha$ for $j = i - k \bmod l_\alpha$ and $j' = i' - k \bmod l_\alpha$. Hence, Lemma 4.15 and Corollary 4.16 yield that $\alpha_j \neq \alpha_{j'}$ as well, which is a contradiction, because $r \nmid k$. Thus, $r' \neq l_\alpha - 2$, which implies $r' \in [2, l_\alpha - 3]$.

Since $m \in [2, l_\alpha - 3]$ and $r' \in [2, l_\alpha - 3]$, there exists a $q \in [1, m - 1]$ such that $q + r' \in [m + 1, l_\alpha - 2]$. Let $i = qk \bmod m_\alpha$ and $i' = i + r \bmod m_\alpha$. Since $r'k \equiv r \pmod{k}$, it follows that $i' = (q + r')\,k \bmod l_\alpha$. Therefore, Lemma 4.15 and Corollary 4.16 imply $\alpha_i \neq \alpha_{i'}$, and hence $r | i, i'$. The definitions of i and i' give $j = (q + 1)\,k \bmod l_\alpha$ and $j' = (q + r' + 1)\,k \bmod l_\alpha$ for $j = i + k \bmod l_\alpha$ and $j' = i' + k \bmod l_\alpha$. Hence, Lemma 4.15 and Corollary 4.16 yield that $\alpha_j \neq \alpha_{j'}$ as well, which is a contradiction, because $r \nmid k$. Thus, there is no $n \in [1, l_\alpha]$ for which $h_\alpha(n) < nh_\alpha(1)$, which completes the proof. □

Lemmas 4.14 and 4.17 yield the following corollary.

Corollary 4.18. *$h_\alpha(n) = nh_\alpha(1)$ for each $n \in [1, l_\alpha]$.*

Lemma 4.19. *$h(n) = nh(1)$ for each $n \in [1, t]$.*

Proof. In case $l_\beta = 1$ we have $(b_i)_{i=1}^{t-1} = \boldsymbol{\alpha}$, $t = l_\alpha$, and hence $h(n) = h_\alpha(n)$. Thus, in case $l_\beta = 1$, Corollary 4.18 implies $h(n) = nh(1)$ for $n \in [1, t]$. Now suppose that $l_\beta > 1$. Since

$$\liminf_{n\to\infty} \frac{h_\beta(n)}{n} = \limsup_{n\to\infty} \frac{h_\beta(n)}{n} = h_\beta(1),$$

[13, Theorem 3.3] yields that $h_\beta(n) = nh_\beta(1)$ for $n \in \left[1, l_\beta\right]$. Therefore, Lemma 4.3 yields that $h(n) = nh(1)$ for each $n \in \left[1, l_\beta\right]$. Moreover, since $h_\beta\left(l_\beta\right) = l_\beta h_\beta(1)$, [13, Corollary 5.2] and [13, Lemma 5.5] show that the cyclicity of the sequence $\boldsymbol{\beta}$ equals l_β. Hence, Corollary 4.18 and Lemma 4.4 yield that $h(n) = nh(1)$ for each $n \in \left[l_\beta, t\right]$. Thus, $h(n) = nh(1)$ for each $n \in [1, t]$, which completes the proof. □

The lemma concludes the proof that the conditions in Theorem 3.5 are sufficient for $\underline{L} = \bar{L} > 0$, and thereby the proof of Theorem 3.5.

5. Proof of Proposition 3.10 and Theorem 3.12

5.1. *Proof of Proposition 3.10*

In this subsection we prove Proposition 3.10. Thus, let d, B, $(\gamma_i)_{i=1}^{d}$, and $(\delta_i)_{i=1}^{d-1}$ be as in the proposition, and suppose throughout the subsection that G is abelian.

Lemma 5.1. *Let $k_1, k_2 \in [1, d-1]$, and $x, y \in B$. If $a_{k_2} a_{k_1}^{-1} = xy^{-1}$, then $k_1 = k_2$.*

Proof. Since $|B| = g/d$, equality (3.4) yields that the sets $\gamma_i B$, $1 \le i \le d$, are disjoint. Therefore, for each $\alpha \in G$ there exist unique $i \in [1, d]$ and $\beta \in B$ such that $\alpha = \gamma_i \beta$. Suppose $\gamma_{k_2}\gamma_{k_1}^{-1} = xy^{-1}$. Then $\gamma_{k_2} y = \gamma_{k_1} x$ where $x, y \in B$, and therefore $k_2 = k_1$ and $y = x$. □

Lemma 5.2. *Let $k \in [1, d-1]$. If the subsequences $(\delta_i)_{i=1}^{k-1}$ and $(\delta_i)_{i=k+1}^{d-1}$ are palindromic, then*

$$\bigcup_{i=1}^{k} \gamma_i B = \gamma_d^{-1}\delta_k^{-1} \cdot \bigcup_{i=1}^{k} \gamma_i B, \tag{5.1}$$

$$\bigcup_{i=k+1}^{d} \gamma_i B = \gamma_d^{-1}\delta_k^{-1} \cdot \bigcup_{i=k+1}^{d} \gamma_i B. \tag{5.2}$$

Proof. Let $j \in [k+1, d]$ and $x \in B$. Put $\omega = \gamma_d^{-1}\delta_k^{-1}\gamma_j x$. Since $\omega \in G$, by (3.4) there exist an $m \in [1, d]$ and some $y \in B$ such that $\omega = \gamma_m y$. Suppose that $m \le k$. Obviously,

$$\gamma_{k+1+(d-j)} = \gamma_m \cdot \prod_{i=m}^{k-1} \delta_i \cdot \delta_k \cdot \prod_{i=k+1}^{k+(d-j)} \delta_i.$$

Since the sequence $(\delta_i)_{i=k+1}^{d-1}$ is palindromic and G is abelian

$$\prod_{i=k+1}^{k+(d-j)} \delta_i = \prod_{i=j}^{d-1} \delta_i = \gamma_d \gamma_j^{-1}.$$

The sequence $(\delta_i)_{i=1}^{k-1}$ is also palindromic, and hence

$$\prod_{i=m}^{k-1} \delta_i = \prod_{i=1}^{k-m} \delta_i = \gamma_{k-m+1}\gamma_1^{-1} = \gamma_{k-m+1}.$$

(This is also true for the case $k - m = 0$ because $\gamma_1 = e$.) Thus, $\gamma_{k+1+(d-j)} = \gamma_m \gamma_{k-m+1}\delta_k\gamma_d\gamma_j^{-1}$. Since $\gamma_m = \omega y^{-1}$, it follows that $\gamma_{k+1+(d-j)} = \gamma_{k-m+1} x y^{-1}$. Therefore, $\gamma_{k+1+(d-j)}\gamma_{k-m+1}^{-1} = xy^{-1}$, where $x, y \in B$, and hence Lemma 5.1 yields $k + 1 + (d-j) = k - m + 1$. Thus, $d - j = -m$, which is a contradiction as $d - j \ge 0$ and $-m \le -1$. Therefore $m \ge k+1$. This actually means that

$$\left(\gamma_d^{-1}\delta_k^{-1}\right)\gamma_j x \in \bigcup_{i=k+1}^{d} \gamma_i B$$

for each $j \in [k+1, d]$ and $x \in B$. Thus,

$$\gamma_d^{-1}\delta_k^{-1} \bigcup_{i=k+1}^{d} \gamma_i B \subseteq \bigcup_{i=k+1}^{d} \gamma_i B.$$

On the other hand, the set on the right-hand side is of order $(d-k)\,g/d$ as well as the set on the left-hand side. Therefore, (5.2) is valid.

Now, suppose that

$$\gamma_d^{-1}\delta_k^{-1} \bigcup_{i=1}^{k} \gamma_i B \nsubseteq \bigcup_{i=1}^{k} \gamma_i B.$$

Then, there exist some $j \in [1, k]$ and $x \in B$ such that

$$\left(\gamma_d^{-1}\delta_k^{-1}\right) \gamma_j x \notin \bigcup_{i=1}^{k} \gamma_i B. \tag{5.3}$$

Equality (3.4) implies the existence of an $m \in [1, d]$ and some $y \in B$ such that $\left(\gamma_d^{-1}\delta_k^{-1}\right)\gamma_j x = \gamma_m y$. Obviously, (5.3) gives $m \geq k+1$. Therefore, (5.2) yield the existence of an $i \in [k+1, d]$ and some $x' \in B$ such that $\gamma_m y = \left(\gamma_d^{-1}\delta_k^{-1}\right)\gamma_i x'$. Thus, $\gamma_j x = \gamma_i x'$, and hence $\gamma_j \gamma_i^{-1} = x' x^{-1}$. Lemma 5.1 implies $j = i$, which is a contradiction as $j \leq k$ and $i \geq k+1$. Thus

$$\left(\gamma_d^{-1}\delta_k^{-1}\right) \bigcup_{i=1}^{k} \gamma_i B \subseteq \bigcup_{i=1}^{k} \gamma_i B.$$

Moreover, the set on the right-hand side is of order kg/d, as is the set on the left-hand side. Thus, (5.2) holds as well. □

Lemma 5.3. *Let $k_1, k_2 \in [1, d_1 - 1]$, such that $k_1 < k_2$. Suppose that $|\{\delta_i : 1 \leq i \leq d-1\}| = 2$, $\delta_{k_1} \neq \delta_{k_2}$, and the subsequences $(\delta_i)_{i=1}^{k_j - 1}$ and $(\delta_i)_{i=k_j+1}^{d-1}$ are palindromic for $j = 1, 2$. Then, $\gamma_d B = \delta_{k_2}^{-1} B$.*

Proof. Let $x \in B$. Since the subsequences $(\delta_i)_{i=1}^{k_1-1}$ and $(\delta_i)_{i=k_1+1}^{d-1}$ are palindromic, Lemma 5.2 show that

$$\bigcup_{i=1}^{k_1} \gamma_i B = \left(\gamma_d^{-1}\delta_{k_1}^{-1}\right) \bigcup_{i=1}^{k_1} \gamma_i B.$$

Since $\gamma_1 = 0$, the equality above implies that there exist some $j \in [1, k_1]$ and $y \in B$ such that $\gamma_j y = \left(\gamma_d \delta_{k_1}\right) x$. Moreover, the subsequences $(\delta_i)_{i=1}^{k_2-1}$ and $(\delta_i)_{i=k_2+1}^{d-1}$ are also palindromic, and hence Lemma 5.2 gives

$$\bigcup_{i=1}^{k_2} \gamma_i B = \left(\gamma_d^{-1}\delta_{k_2}^{-1}\right) \bigcup_{i=1}^{k_2} \gamma_i B.$$

Since $1 \leq j \leq k_1$ and $k_1 < k_2$, we obviously have $1 \leq j \leq k_2$. Therefore, the equality above yields the existence of an $m \in [1, k_2]$ and some $y' \in B$ such that $\gamma_m y' = \left(\gamma_d^{-1}\delta_{k_2}^{-1}\right)\gamma_j y$. Since $\gamma_j y = \left(\gamma_d \delta_{k_1}\right) x$, this implies that $\gamma_m y' = \delta_{k_1}\delta_{k_2}^{-1} x$. Thus,

$$\left(\delta_{k_1}\delta_{k_2}^{-1}\right) B \subseteq \bigcup_{i=1}^{k_2} \gamma_i B. \tag{5.4}$$

Now, let $x \in B$. Since the subsequences $(b_i)_{i=1}^{k_2-1}$ and $(b_i)_{i=k_2+1}^{d-1}$ are palindromic, Lemma 5.2 shows that

$$\bigcup_{i=k_2+1}^{d} \gamma_i B = \left(\gamma_d^{-1}\delta_{k_2}^{-1}\right) \bigcup_{i=k_2+1}^{d} \gamma_i B.$$

Therefore, there exist an $m \in [k_2+1, d]$ and some $y \in B$ such that $\gamma_m y = \left(\gamma_d^{-1}\delta_{k_2}^{-1}\right)\gamma_d x$. Hence, $\gamma_m y = \delta_{k_2}^{-1} x$. Suppose that $m < d$. Then $\gamma_{m+1} y = \delta_m \gamma_m y$, and hence $\gamma_{m+1} y = \delta_m \delta_{k_2}^{-1} x$. Since $|\{\delta_i : 1 \leq i \leq d-1\}| = 2$ and $\delta_{k_1} \neq \delta_{k_2}$, we have either $\delta_m = \delta_{k_1}$ or $\delta_m = \delta_{k_2}$. If $\delta_m = \delta_{k_2}$, then $\gamma_{m+1} y = x = \gamma_1 x$, which contradicts Lemma 5.1. If $\delta_m = \delta_{k_1}$, then $\gamma_{m+1} y = \delta_{k_1}\delta_{k_2}^{-1} x$. Since $\gamma_1 = 0$, equality (5.4) yields that there exist an $m' \in [1, k_2]$ and some $y' \in B$ such that $\gamma_{m'} y' = \delta_{k_1}\delta_{k_2}^{-1} x$. Therefore, $\gamma_{m+1}\gamma_{m'}^{-1} = y' y^{-1}$, which contradicts Lemma 5.1. Thus, $m = d$, and hence $\delta_{k_2}^{-1} x \in \gamma_d B$. Therefore $\delta_{k_2}^{-1} B \subseteq \gamma_d B$. As $\left|\delta_{k_2}^{-1} B\right| = |\gamma_d B|$, we may deduce that $\delta_{k_2}^{-1} B = \gamma_d B$, which completes the proof. □

Put $\gamma_1' = 0$ and $\gamma_j' = \prod_{i=1}^{j-1} \delta_{d-i}$ for $j \in [2, d]$. The following lemma proves that the sequence $\left(\gamma_i'\right)_{i=1}^{d}$ admits similar properties to the sequence $(\gamma_i)_{i=1}^{d}$.

Lemma 5.4. *We have:*

$$\bigcup_{i=1}^{d} \gamma_i' B = G. \tag{5.5}$$

Proof. Since $|B| = g/d$, equality (5.5) holds if and only if for any two distinct $j_1, j_2 \in [1, d]$ there are no $x, y \in B$ such that $\gamma_{j_2}' \left(\gamma_{j_1}'\right)^{-1} \neq xy^{-1}$. Suppose that, for some distinct $j_1, j_2 \in [1, d]$, such that $j_1 < j_2$, there exist $x, y \in B$ such that $\gamma_{j_2}' \left(\gamma_{j_1}'\right)^{-1} = xy^{-1}$. The definitions of γ_{j_2}' and γ_{j_1}' imply $\prod_{i=j_1}^{j_2-1} \delta_{d-i} = xy^{-1}$. Therefore,

$$\gamma_{d-j_1+1}\gamma_{d-j_2+1}^{-1} = \prod_{i=d-j_2+1}^{d-j_1} \delta_i = \prod_{i=j_1}^{j_2-1} \delta_{d-i} = xy^{-1}.$$

Since $j_1 < j_2$, we have $d - j_1 + 1 > d - j_2 + 1$, which contradicts Lemma 5.1. Thus, for any two distinct $j_1, j_2 \in [1, d]$ there are no $x, y \in B$ such that $\gamma_{j_2}' \left(\gamma_{j_1}'\right)^{-1} = xy^{-1}$, what yields (5.5). $\square$

Put $\delta_i' = \gamma_{i+1}' \left(\gamma_i'\right)^{-1}$ for $i \in [1, d-1]$. Note that there is complete symmetry between the sequences $(\gamma_i)_{i=1}^{d}$ and $(\delta_i)_{i=1}^{d-1}$, and the sequences $\left(\gamma_i'\right)_{i=1}^{d}$ and $\left(\delta_i'\right)_{i=1}^{d-1}$.

Proof of Proposition 3.10. First of all, Lemma 5.3 implies that $\gamma_d B = \delta_{k_2}^{-1} B$. On the other hand, Lemma 5.4 yields a complete symmetry between the sequences $(\gamma_i)_{i=1}^{d}$ and $(\delta_i)_{i=1}^{d-1}$, and the sequences $\left(\gamma_i'\right)_{i=1}^{d}$ and $\left(\delta_i'\right)_{i=1}^{d-1}$. Note that $\delta_i' = \delta_{d-i}$ for each $i \in [1, d-1]$. Therefore, $1 \le d - k_2 < d - k_1 \le d - 1$, $\delta_{d-k_2}' \neq \delta_{d-k_1}'$, $\left|\left\{\delta_i' : 1 \le i \le d-1\right\}\right| = 2$, and the subsequences $\left(\delta_i'\right)_{i=1}^{d-k_j-1}$ and $\left(\delta_i'\right)_{i=d-k_j+1}^{d-1}$ are palindromic for $j = 1, 2$. Thus, we may apply Lemma 5.3 also to the sequences $\left(\gamma_i'\right)_{i=1}^{d}$ and $\left(\delta_i'\right)_{i=1}^{d-1}$, which yields $\gamma_d' B = \left(\delta_{d-k_1}'\right)^{-1} B$. The definitions give $\gamma_d' = \prod_{i=1}^{d-1} \delta_{d-i} = \gamma_d$, and $\delta_{d-k_1}' = \delta_{k_1}$. Therefore, $\gamma_d B = \delta_{k_1}^{-1} B$. Thus, $\delta_{k_1}^{-1} B = \delta_{k_2}^{-1} B$, which yields $B = \left(\delta_{k_2}^{-1}\delta_{k_1}\right) B$, as required. $\square$

5.2. Proof of Theorem 3.12

Lemma 5.5. *Let $(\alpha_i)_{i=1}^{m-1}$ be an appropriate sequence, and let k and k' the integers out of the definition of the appropriate sequence. Then $|\{\alpha_i : 1 \le i \le m-1\}| = 2$, and for $k_1 = k'$ and $k_2 = \left(k' + k\right) \bmod m$, the subsequences $(\alpha_i)_{i=1}^{k_j-1}$ and $(\alpha_i)_{i=k_j+1}^{m-1}$ are palindromic for $j = 1, 2$.*

Proof. Since $\gcd(k, m) = 1$, there exists some $j' \in [0, m-1]$ such that $j'k \equiv k' \pmod m$. Since $k' \neq m - k, 0$ we have $1 \le j' \le m - 2$, and Lemma 4.15 stems that for $i, j \in [1, m-1]$ such that $i = jk \bmod m$, if $j \in \left[1, j'\right]$ then $\alpha_i = \alpha_k$, and if $j \in \left[j' + 1, m - 1\right]$ then $\alpha_i = \alpha_{m-k}$. Since $\alpha_{j'k \bmod m} \neq \alpha_{(j'+1)k \bmod m}$, it follows that $|\{\alpha_i : 1 \le i \le m-1\}| = 2$.

Let $i \in \left[1, k' - 1\right]$ and $j \in [1, m-1]$ be such that $i = jk \bmod m$. Suppose that $\alpha_i = \alpha_k$, and hence $j \in \left[1, j'\right]$. Therefore $j' - j \in [1, j]$, and the equality $k' - i = j'k - jk \bmod m$ gives $\alpha_{k'-i} = \alpha_k$. Thus, if $\alpha_i = \alpha_k$, then $\alpha_{k'-i} = \alpha_k$ as well. Now, suppose that $\alpha_i = \alpha_{m-k}$ and $\alpha_{k'-i} = \alpha_k$. We just proved that, since $\alpha_{k'-i} = \alpha_k$, we have $\alpha_i = \alpha_{k'-(k'-i)} = \alpha_k$, which is a contradiction. Thus, if $\alpha_i = \alpha_{m-k}$, then $\alpha_{k'-i} = \alpha_{m-k}$. Consequently, $\alpha_i = \alpha_{k'-i}$, which means that the subsequence $(\alpha_i)_{i=1}^{k'-1}$ is palindromic.

Let $(\beta_i)_{i=1}^{m-1}$ be the sequence defined by $\beta_i = \alpha_{m-i}$. Since $(\alpha_i)_{i=1}^{m-1}$ is appropriate, the definition of $(\beta_i)_{i=1}^{m-1}$ yields

$$\beta_i = \beta_{(i+k) \bmod m}, \quad i \in [1, m-1] \setminus \{m-k, k''\},$$
$$\beta_{k''} \neq \beta_{(k''+k) \bmod m},$$

where $k'' = \left(-k' - k\right) \bmod m$. Note that $k'' \in [1, m-1] \setminus \{m-k\}$, and therefore $(\beta_i)_{i=1}^{m-1}$ is an appropriate sequence as well, which makes the sequence $(\beta_i)_{i=1}^{k''-1}$ be palindromic. Thus, $(\alpha_i)_{i=k_2+1}^{m-1}$ is palindromic.

By the same token, the subsequences $(\alpha_i)_{i=1}^{k_2-1}$ and $(\alpha_i)_{i=k_1+1}^{m-1}$ are palindromic as well, which completes the proof. $\square$

Now, we are ready to prove Theorem 3.12.

Proof of Theorem 3.12. Suppose that G is cyclic of a prime order, $t = g$, $(a_i)_{i=1}^{g}$ is a permutation and

$$\liminf_{n\to\infty} \frac{f(n)}{n} = \limsup_{n\to\infty} \frac{f(n)}{n}.$$

Therefore, by [13, Theorem 3.1], $\underline{L} = \bar{L}$ for $(b_i)_1^{g-1} = \left(a_i^{-1}a_{i+1}\right)_{i=1}^{g-1}$ and $\varphi(\gamma) = a_g^{-1}\gamma$. Since $(a_i)_{i=1}^{g}$ is a permutation and $a_1 = e$, we have $a_g \neq e$, and hence $\varphi(b_1) = a_g^{-1}b_1 \neq b_1$. Therefore, according to [13, Theorem 3.3] we have $\underline{L}, \bar{L} \neq 0$, and hence Proposition 3.1 yields that $\underline{L} = \bar{L} = h(1)$. Therefore, the sequence $(b_i)_1^{g-1}$ satisfies the conditions of Theorem 3.5, and first of all there exist an appropriate sequence $\boldsymbol{\alpha}$, and another sequence $\boldsymbol{\beta}$ for which $(b_i)_{i=1}^{g-1} = \boldsymbol{\alpha} \odot \boldsymbol{\beta}$. Since g is prime, it follows that $\boldsymbol{\beta}$ is the empty sequence, and hence $(b_i)_{i=1}^{g-1} = \boldsymbol{\alpha}$, which makes $(b_i)_{i=1}^{g-1}$ be appropriate. Therefore, Lemma 5.5 implies that $|\{b_i : 1 \leq i \leq g-1\}| = 2$, and for two different integers, k_1 and k_2, the subsequences $(\alpha_i)_{i=1}^{k_j-1}$ and $(\alpha_i)_{i=k_j+1}^{g-1}$ are palindromic for $j = 1, 2$. Since $(a_i)_{i=1}^{g}$ is a permutation, for $B = \{e\}$ we have $\bigcup_{i=1}^{g} a_iB = G$. Thus, Proposition 3.10 implies that $B = \left(b_{k_2}^{-1}b_{k_1}\right)B$. Since $b_{k_2} \neq b_{k_1}$, we have $b_{k_2}^{-1}b_{k_1} \neq e$, and therefore it follows that $B \neq \{e\}$, which is a contradiction. Thus,

$$\liminf_{n\to\infty} \frac{f(n)}{n} < \limsup_{n\to\infty} \frac{f(n)}{n},$$

which completes the proof. □

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